Indag. Mathem., N.S., 11 (2), 219–237 June 26, 2000

Embeddability of quadratic forms in Pfister forms*

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Oleg Izhboldin died unexpectedly in Paris on April 17, 2000.
The first author dedicates this paper to the memory of Oleg who was a very talented and productive mathematician and a friend who will be sadly missed.

Communicated by Prof. T.A. Springer at the meeting of December 20, 1999

ABSTRACT

Let F be a field of characteristic $\neq 2$ and let q be an anisotropic quadratic form over F. The form q is called m-embeddable if q is similar to a subform of an anisotropic m-fold Pfister form. This property can be expressed in terms of Milnor K-theory. By a theorem of Elman-Lam, the form q is m-embeddable if and only if the kernel of the homomorphism $K_*^M(F)/2 \to K_*^M(F(q))/2$ contains a nontrivial symbol of degree m. Let $m(q)$ (resp. $m_{ext}(q)$; resp. $m_{ptr}(q)$) be the smallest integer m such that q is m-embeddable over F (resp. over an extension of F; resp. over a purely transcendental extension of F). We study the possible values of the invariants $m(q)$ and $m_{ext}(q)$ for forms of a given dimension d. We also prove that the invariant m_{ptr} depends only on $m(q)$ and $m_{ext}(q)$, more precisely, $m_{ptr}(q) = \min\{m(q), m_{ext}(q) + 1\}$. In particular, this implies that any form of dimension $\leq 2^n + 1$ is $(n+2)$-embeddable over a suitable purely transcendental extension of the field F. As an application, we show that for certain 'generic' quadratic forms q of dimension d, each system of homogeneous elements which generate the kernel of the map $K_*^M(F)/2 \to K_*^M(F(q))/2$ will necessarily contain elements of degree $d-1$ and of degree $\leq \log_2(d-1) + 2$.

Let F be a field of characteristic $\neq 2$. An n-fold Pfister form over F is a form of the type $\langle 1, -a_1\rangle \otimes \cdots \otimes \langle 1, -a_n\rangle$, $a_i \in F^*$, and we write $\langle\langle a_1, \cdots, a_n\rangle\rangle$ for short. Let P_nF (resp. GP_nF) denote the set of all forms over F which are isometric (resp. similar) to n-fold Pfister forms over F, and let PF (resp. GPF) denote the set of all forms over F which are isometric (resp. similar) to Pfister forms over F.

* The collaboration on this paper has been supported in part by TMR Network ERB FMRX CT-97-0107 'Algebraic K-Theory, Linear Algebraic Groups and Related Structures.' The second author gratefully acknowledges the financial support by the Alexander von Humboldt Foundation.

A form φ over F is called embeddable if there exists an anisotropic $\pi \in GPF$ such that $\pi \cong \varphi \perp \tau$ for a suitable form τ over F, i.e., φ is a subform of π. We will also write $\varphi \subset \pi$ for short. The form φ is called conservative if $W(F(\varphi)/F) \neq 0$ (here, $F(\varphi)$ denotes the function field of φ and $W(F(\varphi)/F)$ denotes the kernel of the ring homomorphism $WF \to WF(\varphi)$ between the Witt rings of F and $F(\varphi)$ which is induced by scalar extension $\psi \mapsto \psi \otimes F(\varphi) = \psi_{F(\varphi)}$). Note that if φ is conservative then necessarily φ is anisotropic, for if φ is isotropic, then it is well-known that $F(\varphi)/F$ is purely transcendental, and anisotropic forms stay anisotropic over purely transcendental extensions, hence $W(F(\varphi)/F) = 0$.

To our knowledge, it is still an open problem whether a conservative form is always embeddable. There are no known counterexamples. This question has been investigated by Gentile and Shapiro [GS] and by Fitzgerald [Fi1], [Fi2].

Fitzgerald [Fi1, Theorem 4.5] has shown that if φ is conservative then there exists a purely transcendental extension L of F of finite transcendence degree such that φ_L is embeddable. It turns out that the assumption on φ being conservative is not needed and that anisotropy alone suffices. We will give a quick and elementary proof in the first section.

If one allows arbitrary field extensions, it was shown by the first author [Ho] that if $\dim \varphi \leq 2^n + 1$ then there exists a field extension K/F such that φ is embeddable into an $(n+1)$-fold Pfister form over K (if $\dim \varphi \leq 2^n$ then one can even find a unirational K). In view of this result and the fact that embeddability can always be achieved over suitable purely transcendental extensions, it becomes a natural problem to determine the smallest m such that there exists an extension L/F (resp. a purely transcendental extension L/F) over which φ is embeddable into an m-fold Pfister form. These smallest values m depend in general on the type of the field extensions we consider, so for each class of field extensions we can in such a way attach an invariant (the m from above) to a given anisotropic form. In the second section, we will obtain certain bounds for these invariants and we will also investigate how these invariants relate to each other. For forms of small dimension the possible values of these invariants will be determined explicitly.

In the third section, we consider stable Pfister neighbors. These are anisotropic forms which become anisotropic Pfister neighbors over some field extension. In particular, to any given $d \in \mathbb{N}$ we will determine the set of $k \in \mathbb{N}$ such that there exists a field F with a stable Pfister neighbor q of dimension d such that q embeds over F in some anisotropic k-fold Pfister form, but not in any anisotropic Pfister form of degree $< k$.

In the fourth section, we derive partial results of that type also for forms which are not stable Pfister neighbors.

In the last section, we give an application of our results concerning the graded ring $K_*^M F/2$ in Milnor K-theory. We consider the kernel of the restriction map induced by passing from F to the function field $F(q)$, where q is an anisotropic form of dimension $2^{n-1} < d = \dim q \leq 2^n$. As an ideal, this kernel can be generated by a system of homogeneous elements. We show that in general this system will necessarily contain elements of degree $\leq n+1$ and of degree $d-1$.

(If we assume recent results of Voevodsky and Orlov-Vishik-Voevodsky, then we can actually show that $\leq n$ is impossible in our example.) This indicates that a simple description of this kernel in terms of a system of homogeneous generators seems rather unlikely in general.

For the proofs we need only basic properties of quadratic forms and some well-known results concerning the behaviour of quadratic forms under purely transcendental extensions, all of which can be found in Lam's book [Lam], most notably Chapter 9. For all undefined notations and terminology, we also refer the reader to Lam's book. The basic reference for the K-theoretic part (Section 5) is Milnor's original article [Mi].

1. EMBEDDABILITY OF QUADRATIC FORMS OVER PURELY TRANSCENDENTAL FIELD EXTENSIONS

Theorem 1.1. *Let φ be an anisotropic form over F. Then there exists a purely transcendental extension L of F of finite degree such that φ_L is embeddable.*

Proof. To begin with, let π be any anisotropic Pfister form over F. Let η be a common subform of φ and π of maximal dimension and write $\varphi \cong \eta \perp \varphi'$ and $\pi \cong \eta \perp \pi'$ for suitable forms φ' and π' over F. Let $s := \dim \varphi'$ and $r := \dim \pi'$. If $s = 0$ then φ is already a subform of π and we are done.

So suppose $s > 0$. Let $\psi := \pi' \perp -\varphi'$. Let $K = F(x_1, \cdots, x_r, y_1, \cdots, y_s)$ be the rational function field in the $r + s$ variables x_i and y_j. Let $X = (x_1, \cdots, x_r)$ and $Y = (y_1, \cdots, y_s)$ so that we have $\psi(X, Y) = \pi'(X) - \varphi'(Y)$. Consider $\langle 1, -\psi(X, Y)\rangle \otimes \pi_K$ which is clearly a Pfister form over K and thus either hyperbolic or anisotropic.

If $\langle 1, -\psi(X, Y)\rangle \otimes \pi_K$ is hyperbolic then $\pi_K \cong \psi(X, Y)\pi_K$. Since π represents 1, it follows that π_K represents $\psi(X, Y)$. By the Third Representation Theorem [Lam, Chapter IX, 2.8], there exists a form γ over F such that $\pi \cong \psi \perp \gamma$. Since we have also $\pi \cong \eta \perp \pi'$ and $\psi \cong \pi' \perp -\varphi'$, we obtain by Witt cancellation that $\eta \cong -\varphi' \perp \gamma$ and thus $\varphi \cong -\varphi' \perp \gamma \perp \varphi'$, which implies that φ is isotropic as $s = \dim \varphi' > 0$, a contradiction.

Therefore, $\langle 1, -\psi(X, Y)\rangle \otimes \pi_K$ is anisotropic. For the Witt index of $\pi \perp -\varphi$ we have by definition of η that $i_W(\pi \perp -\varphi) = \dim \eta$. We now show that, over K, $i_W(\langle 1, -\psi(X, Y)\rangle \otimes \pi_K \perp -\varphi_K) > \dim \eta$, i.e. the anisotropic forms φ_K and $\langle 1, -\psi(X, Y)\rangle \otimes \pi_K \in PK$ have a subform of dimension $> \dim \eta$ in common. Once this is shown, it becomes obvious that by repeating this procedure we will eventually reach a purely transcendental extension L/F of finite degree over which there exists an anisotropic Pfister form ρ such that $i_W(\rho \perp -\varphi_L) \geq \dim \varphi$, which in turn readily implies that φ_L is a subform of ρ, and the theorem follows.

Now

$$\langle 1, -\psi(X, Y)\rangle \otimes \pi_K \perp -\varphi_K \cong \eta_K \perp \pi'_K \perp -\psi(X, Y)\pi_K \perp -\eta_K \perp -\varphi'_K$$

and it clearly suffices to show that $\pi'_K \perp -\psi(X, Y)\pi_K \perp -\varphi'_K$ is isotropic. But over K, π'_K represents $\pi'(X)$, $-\varphi'_K$ represents $-\varphi'(Y)$, and $-\psi(X,Y)\pi_K$ represents $-\psi(X,Y) = -\pi'(X) + \varphi'(Y)$ because π represents 1. Hence, $\pi'_K \perp -\psi(X,Y)\pi_K \perp -\varphi'_K$ is isotropic and the theorem follows. $\square$

If $\dim\varphi = n$ and if we assume, after scaling, that φ represents 1, then by starting with the 0-fold Pfister form $\langle 1\rangle$, we see that the above construction yields a purely transcendental extension over which φ is embeddable into a Pfister form of dimension $\leq 2^{n-1}$. Although the construction as such is quite simple, the bound on the dimension of the Pfister form is far from being the best possible in the general case. In the following section, we will apply more sophisticated arguments which will then lead to the best possible bound.

It should be remarked that A. Vishik [Vi] also proved the above theorem independently. His proof is slightly different as his main motivation is to construct certain 'generic' symbols in Milnor K-theory, but he also uses the Third Representation Theorem in a way similar to ours.

2. EMBEDDABILITY INVARIANTS

Definition 2.1. Let q be an anisotropic form over F and K be a field extension of F. We say that q is *n-embeddable over K* if there exists an anisotropic $\pi \in GP_nK$ such that $q_K \subset \pi$.

Let $\mathcal{F}(F)$ be a class of field extensions of F containing F. We will write $\mathcal{F}$ for short if no confusion regarding the base field can arise. If there is no K in $\mathcal{F}$ such that q is embeddable over K, we put $m_{\mathcal{F}}(q) := \infty$. Otherwise, we define

$$m_{\mathcal{F}}(q) := \min\{n \in \mathbb{N} \mid \exists K \in \mathcal{F} : q \text{ is } n\text{-embeddable over } K\}.$$

In this paper, we will consider the following classes of field extensions:

- $\mathcal{F} = \{F\}$, the class consisting only of the base field F. The corresponding invariant will be denoted by $m(q)$ instead of $m_{\mathcal{F}}(q)$.
- $\mathcal{F} = \mathcal{EXT}$, the class consisting of *all* field extensions of F. The corresponding invariant will be denoted by $m_{ext}(q)$.
- $\mathcal{F} = \mathcal{FGS}$, the class consisting of all finitely generated separable field extensions of F. The corresponding invariant will be denoted by $m_{fgs}(q)$.
- $\mathcal{F} = \mathcal{PTR}$, the class consisting of all finitely generated purely transcendental field extensions of F. The corresponding invariant will be denoted by $m_{ptr}(q)$.

Lemma 2.2. *Let q be an anisotropic form over F and let K be any field extension of F over which q stays anisotropic. Denote by $\mathcal{F}(L)$ and $\mathcal{G}(L)$ classes of field extensions of a fixed base field L with $\mathcal{F}(L) \subset \mathcal{G}(L)$. Then the following holds.*

(i) $m_{\mathcal{F}(F)}(q) \geq m_{\mathcal{G}(F)}(q)$. *In particular,* $m(q) \geq m_{ptr}(q) \geq m_{fgs}(q) \geq m_{ext}(q)$.

(ii) *Suppose that $\mathcal{F}(K) \subset \mathcal{F}(F)$. Then* $m_{\mathcal{F}(K)}(q_K) \geq m_{\mathcal{F}(F)}(q)$.

(iii) *If K/F is purely transcendental, then* $m(q_K) \leq m(q)$, $m_{ptr}(q_K) = m_{ptr}(q)$, $m_{ext}(q_K) = m_{ext}(q)$, *and* $m_{fgs}(q_K) = m_{fgs}(q)$.

Proof. (i) and (ii) are obvious.

(iii) If $m(q)$ is finite and $\pi \in GPF$ is anisotropic such that $q \subset \pi$, then π_K is anisotropic as K/F is purely transcendental, and $q_K \subset \pi_K$. This implies $m(q_K) \leq m(q)$. The fact that $m_{ptr}(q_K) = m_{ptr}(q)$ is trivial. To prove the last two equalities, we may clearly assume that $m_{ext}(q)$ (resp. $m_{fgs}(q)) = n < \infty$. Let L be a (finitely generated separable) field extension of F such that there exists an anisotropic $\pi \in GP_nL$ with $q_L \subset \pi$. Without loss of generality, we may assume that $K = F(T)$ where $L(T)$ is a purely transcendental field extension of L with transcendence basis T. In particular, $\pi_{L(T)}$ is anisotropic. Now $q_{L(T)} \subset \pi_{L(T)}$ together with (ii) (and the fact that $L(T)$ is again finitely generated separable over K if L/F is finitely generated and separable) immediately yields the desired equalities. □

Lemma 2.3. *Let n be an integer ≥ 0 and let q be an anisotropic form over F with $2^{n-1} < \dim q \leq 2^n + 1$. Then $n + 1 \geq m := m_{ext}(q) \geq n$.*

Furthermore, there exists a tower $F \subset L \subset K$ and some $\pi \in GP_mL$ such that L/F is finitely generated purely transcendental, K/L is finite separable algebraic, π_K is anisotropic and $q_K \subset \pi_K$. In particular, $m_{fgs}(q) = m_{ext}(q)$.

Proof. It is clear that $m_{ext}(q) \geq n$. The fact that $n + 1 \geq m_{ext}(q)$ (which in fact implies equality if $\dim q = 2^n + 1$) follows from [Ho, Theorem 2]. It remains to establish the existence of L, K and π with the properties stated in the lemma. This automatically implies that $m_{ext}(q) = m_{fgs}(q)$.

We may assume that q represents 1, so that if $q_E \subset \psi$ for some field extension E/F and some $\psi \in GP_mE$, we in fact have $\psi \in P_mE$.

So let E/F be any field extension such that there exist an anisotropic $\psi \in P_mE$, $m = m_{ext}(q)$, and a form τ over E with $\psi \cong q_E \perp \tau$. Let us fix a diagonalization of q over F and $a_i, b_j \in E^*$ such that $\psi \cong \langle\langle a_1, \cdots, a_m\rangle\rangle$, $\tau \cong \langle b_1, \cdots, b_r\rangle$ where $r = 2^m - \dim q$. Let $S \subset E$ be the finite set consisting of the a_i's, b_j's, and the coefficients which appear in the transformation matrix associated to the above isometry with respect to the chosen diagonalizations. It is obvious that ψ, τ, and the isometry are already defined over $F(S)$, with ψ being defined by some form in $P_mF(S)$. Thus, by replacing E by $F(S)$, we may assume that E is already finitely generated over F.

Let $T_1, \cdots, T_m$ be independent variables over E, $M = F(T_1, \cdots, T_m)$, $\pi := \langle\langle T_1, \cdots, T_m\rangle\rangle \in P_mM$, and $N = E(T_1, \cdots, T_m)(\sqrt{T_1a_1}, \cdots, \sqrt{T_ma_m})$. Obviously, N/E is purely transcendental, hence ψ_N is anisotropic. Furthermore, $\psi_N \cong \pi_N$. In particular, π_N is anisotropic and $q_N \subset \pi_N$. By general field theory, one can extend the set $\{T_1, \cdots, T_m\}$ to a transcendence basis of N/F so that we get a tower $F \subset M \subset F_t \subset F_s \subset N$ such that F_t/F is purely transcendental, F_s/F_t is separable algebraic, and N/F_s is purely inseparable algebraic.

It is well-known that the map which sends a form $\langle c_1, \cdots, c_n\rangle$ over F_s to the form $\langle c_1, \cdots, c_n\rangle_N$ over the purely inseparable extension N yields a $1-1$ correspondence between isometry classes of quadratic forms over F_s and N which preserves (an)isotropy, subforms, etc., and which is also $1-1$ between GPF_s

and GPN. Thus, we may replace N by F_s and we can conclude that q is already embeddable over F_s into a form in P_mF_s which is defined over M (and hence over F_t) by the m-fold Pfister form π. Note that E/F and therefore also N/F is finitely generated. It follows readily that F_t/F is finitely generated purely transcendental, and F_s/F_t is finite separable algebraic. We put $L = F_t$ and $K = F_s$. $\square$

Lemma 2.4. *Let q be an anisotropic form over F and $m := m_{ext}(q)$. Then there exist a finitely generated purely transcendental field extension E/F and an anisotropic $\pi \in GP_{m+1}E$ such that $q_E \subset \pi$. In particular, $m_{ptr}(q) \leq m_{ext}(q) + 1$.*

Proof. After scaling, we may assume that q represents 1. We may also assume that $\dim q \geq 4$ as forms of dimension ≤ 3 are always Pfister neighbors and there is nothing to show.

By Lemma 2.3, there exist a finitely generated purely transcendental field extension L/F, an anisotropic $\tau \in P_mL$, and a finite separable field extension K/L such that τ_K is anisotropic and $q_K \subset \tau_K$.

Since K/L is finite separable, there exists, by general field theory, a primitive element $\theta \in L$ such that $K = L(\theta)$. Let $p(t) \in L[t]$ be the monic irreducible polynomial of θ. We will show that $E = L(t)$ and $\pi := \langle\langle p(t)\rangle\rangle \otimes \tau_E \in P_{m+1}E$ have the desired properties.

Clearly, E/F is finitely generated purely transcendental.

Next, we show that π is anisotropic. Suppose $\pi \cong \langle 1, -p(t)\rangle \otimes \tau_E$ is isotropic and hence hyperbolic. Then $p(t)$ is a similarity factor of τ_E, and by [Lam, Chapter IX, Theorem 3.4] we have that τ_K is hyperbolic, a contradiction. Thus, π is anisotropic.

We finally show that q_E is a subform of π. Since $\dim q \geq 3$, L is algebraically closed in $L(q)$ (see, e.g., [Kn, Proposition 3.6]). Hence, $p(t)$ is also the irreducible monic polynomial of θ over $L(q)$. Since $q_{L(\theta)} \subset \tau_{L(\theta)}$, we obviously have that $\tau_{L(q)(\theta)}$ is isotropic and hence hyperbolic. By [Lam, Chapter IX, Theorem 3.4], it follows this time that $\langle\langle p(t)\rangle\rangle \otimes \tau_{L(q)(t)} = \pi_{L(q)(t)}$ is hyperbolic. With $E = L(t)$ and since $L(q)(t) \cong E(q)$ over L, we have that π is anisotropic over E and hyperbolic over $E(q)$. By the Cassels-Pfister subform theorem and since q and π represent 1, it follows that $q_E \subset \pi$. $\square$

Lemma 2.5. *Let q be an anisotropic quadratic form over F and E/F be a finitely generated purely transcendental field extension. Suppose that $m_{ext}(q_E) = m(q_E)$. Then $m_{ext}(q) = m(q)$.*

Proof. By induction, we may assume that $E = F(t)$. We may also assume that q represents 1 and $\dim q \geq 3$. Let $n = m_{ext}(q)$. Since $m_{ext}(q_E) = m(q_E)$, it follows that $m(q) \geq m(q_E) = m_{ext}(q_E) = m_{ext}(q) = n$ (cf. Lemma 2.2). It suffices to prove that $m(q) \leq n$.

Since $m(q_E) = n$, and since q represents 1, there exists an anisotropic $\pi \in P_nE$ such that $q_E \subset \pi$. As mentioned in the proof of Lemma 2.4, any irreducible polynomial over F stays irreducible over $F(q)$. Let $K = F(q)(t)$. We then obtain the following commutative diagram, where the direct sums range over all monic irreducible polynomials $p \in F[t]$ (notations as in [Lam, p. 265 ff.] resp. [Mi, p. 335 and Lemma 5.7]):

$$\begin{array}{ccccccc} 0 & \longrightarrow & I^nF & \xrightarrow{i} & I^nE & \xrightarrow{\oplus_p \partial_p} & \bigoplus_p I^{n-1}\overline{E}_p & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & I^nK & \xrightarrow{\oplus_p \partial_p} & \bigoplus_p I^{n-1}\overline{K}_p & & \end{array}$$

By [Mi, Lemma 5.7], the upper row in this diagram is an exact sequence.

Let $p \in F[t]$ be a monic irreducible polynomial. Since $\pi \in P_nE$, there exists $\pi_p \in GP_{n-1}\overline{E}_p$ such that $\partial_p(\pi) = \pi_p$ in $W\overline{E}_p$. Then $(\pi_p)_{\overline{K}_p} = \partial_p(\pi)_{\overline{K}_p} = \partial_p(\pi_K) = \partial_p(0) = 0$ in $W\overline{K}_p$ as $\pi_K = 0$. If π_p is anisotropic, it follows from the Cassels-Pfister subform theorem and from the fact that $\overline{K}_p = \overline{E}_p(q)$ (recall that F is algebraically closed in $F(q)$!), that $q_{\overline{E}_p}$ is similar to a subform of the anisotropic form $\pi_p \in GP_{n-1}\overline{E}_p$. Then $m_{ext}(q) \leq n-1$, a contradiction.

Therefore $\partial_p(\pi) = \pi_p = 0$ for all monic irreducible polynomials $p \in F[t]$. Hence $\pi \in \ker(I^nE \to \bigoplus_p I^{n-1}\overline{E}_p) = \text{im}\,(I^nF \to I^nF(t))$. Thus, π as an element in I^nE is defined over F by a form $\tau \in I^nF$. Using well-known properties of purely transcendental extensions applied to $E = F(t)$, one concludes readily that there exists a form $\tau \in P_nF$ such that $\pi \cong \tau_E \in P_nE$, and $q_E \subset \pi$ implies that we have $q \subset \tau$. Hence $m(q) \leq n$. $\square$

Theorem 2.6. *Let q be an anisotropic form over F with $2^{n-1} < \dim q \leq 2^n$. Then $m_{ptr}(q) = \min\{m(q), m_{ext}(q)+1\} \in \{n, n+1, n+2\}$.*

Proof. By Lemmas 2.3 and 2.4, we have $m_{ptr}(q) \in \{m_{ext}(q), m_{ext}(q)+1\} \subset \{n, n+1, n+2\}$.

Suppose first that $m_{ptr}(q) = m_{ext}(q)$. By definition of $m_{ptr}(q)$ and $m(q)$, there exists a finitely generated purely transcendental extension K/F such that $m(q_K) = m_{ptr}(q)$. By Lemma 2.2, $m_{ext}(q) = m_{ext}(q_K)$. All this together implies $m(q_K) = m_{ext}(q_K)$ and thus, by Lemma 2.5, $m(q) = m_{ext}(q)$. Hence, $m_{ptr}(q) = m_{ext}(q) = m(q) = \min\{m(q), m_{ext}(q)+1\}$.

If $m_{ptr}(q) = m_{ext}(q)+1$, then again $m_{ptr}(q) = \min\{m(q), m_{ext}(q)+1\}$ because we always have $m(q) \geq m_{ptr}(q)$. $\square$

The above theorem shows how m_{ptr} depends on m and m_{ext}, and we also know that for anisotropic q with $2^{n-1} < \dim q \leq 2^n$ we have $m_{ext}(q) \in \{n, n+1\}$, where $m_{ext}(q) = n \geq 1$ if $2^{n-1}+1 = \dim q$. One would naturally also like to have more precise information about $m(q)$. We have the following.

Proposition 2.7. *Let $q = q_0 \perp \mu$ be an anisotropic form such that q_0 is a Pfister neighbor. If $\pi \in GPF$ is anisotropic with $q \subset \pi$, then there exists*

$s \leq m(q_0) + \dim \mu$ *and* $\sigma \in GP_sF$ *such that* $q \subset \sigma \subset \pi$. *In particular, if* $m(q)$ *is finite then* $m(q_0) \leq m(q) \leq m(q_0) + \dim \mu$.

Proof. Throughout, we may assume that q_0 represents 1. We use induction on $\dim \mu$. If $\dim \mu = 0$, then $q = q_0$ is a Pfister neighbor of some anisotropic $\sigma \in P_{m(q)}F$. Let $\pi \in GPF$ be anisotropic with $q \subset \pi$. Note that $q \subset \sigma$ and $\pi \in PF$ as all three forms represent 1. Since q becomes isotropic over $F(\sigma)$, it follows that π becomes hyperbolic over $F(\sigma)$. The Cassels-Pfister subform theorem implies $q \subset \sigma \subset \pi$.

So let us now assume that $r = \dim \mu \geq 1$, and write $q \cong q_0 \perp \langle a_1, \cdots, a_r \rangle$. Suppose there exists an anisotropic Pfister form π such that $q \subset \pi$ (recall that q represents 1!). Note that $q_1 := q_0 \perp \langle a_1, \cdots, a_{r-1} \rangle \subset \pi$. By induction hypothesis, there exists $\tau \in P_tF$ such that $q_1 \subset \tau \subset \pi$ for some $t \leq m(q_0) + (r-1)$. Write $\tau \cong q_1 \perp \tau_1$ and $\pi \cong \tau \perp \pi_1$, so that

$$q \cong q_1 \perp \langle a_r \rangle \subset \pi \cong q_1 \perp \tau_1 \perp \pi_1 .$$

By Witt cancellation, we have that $a_r = u + v$ where u (resp. v) is represented by τ_1 (resp. π_1). If $v = 0$ we have that $q \subset \tau$ and we can put $\sigma = \tau$. If $v \neq 0$, then $q \subset \tau \perp \langle v \rangle \subset \pi$, and since $\tau \perp \langle v \rangle$ is a Pfister neighbor of $\tau \otimes \langle\langle -v \rangle\rangle$, it follows from [EL, Theorem 2.7] that $\tau \otimes \langle\langle -v \rangle\rangle \subset \pi$. With $\sigma = \tau \otimes \langle\langle -v \rangle\rangle$, we now get $q \subset \sigma \subset \pi$ and $\sigma \in P_{t+1}F$ with $t + 1 \leq m(q_0) + r$ as desired. $\square$

Remark 2.8. In the previous proposition, one can always take for q_0 any 1-dimensional subform of q so that $m(q_0) = 0$, and in this way we get $m(q) \leq \dim q - 1$, an inequality which is essentially due to Knebusch and which has been 'rediscovered' (stated somewhat differently) by Ahmad and Ohm in [AO, Theorem 2.4]. The proof of our slightly more general statement given above is based on the proof of Knebusch's result which can be found in [Fi3, Lemma 2.1].

For forms of small dimension, we can give more explicit information as for the possible values of m, m_{ext} (and thus also of m_{ptr}). We collect it in the following proposition, omitting the trivial cases where q is a Pfister neighbor, in particular the cases where $\dim q \leq 3$.

Proposition 2.9. *Let q be an anisotropic form over F which is not a Pfister neighbor.*

(i) $\dim q = 4$: *then* $m(q) \in \{3, \infty\}$, $m_{ext}(q) = 2$, *and* $m_{ptr}(q) = 3$.

(ii) $\dim q = 5$: *then* $m(q) \in \{4, \infty\}$, $m_{ext}(q) = 3$, *and* $m_{ptr}(q) = 4$.

(iii) $\dim q = 6, 7,$ *or* 8: *then* $m(q) \in \{4, \cdots, \dim q - 1, \infty\}$, $m_{ext}(q) \in \{3, 4\}$, *and* $m_{ptr}(q) \in \{4, 5\}$. *If q contains a subform in GP_2F, then* $m(q) \neq \dim q - 1$ *(in the case* $\dim q = 6$ *this implies that* $m(q) \in \{4, \infty\}$*).*

Furthermore,

- $m_{ext}(q) = 3$ *iff q does not contain an Albert form as a subform. In this case* $m_{ptr}(q) = 4$. *(An Albert form over F is a form of dimension* 6 *in* I^2F.*)*

- $m_{ext}(q) = 4$ *iff q contains an Albert form as a subform. In this case $m_{ptr}(q) = 4$ or 5 depending on whether $m(q) = 4$ or ≥ 5.*

(iv) $\dim q = 9$: *$m(q) \in \{5, 6, 7, 8, \infty\}$, $m_{ext}(q) = 4$, and $m_{ptr}(q) = 5$.*

Proof. It suffices to verify the statements for $m(q)$ and $m_{ext}(q)$. The results about $m_{ptr}(q)$ then follow from Theorem 2.6. In Remark 2.8, we showed that $m(q) \leq \dim q - 1$ if $m(q)$ is finite.

(i) Suppose that $\dim q = 4$. Let $d = d_{\pm}q$ be the signed discriminant of q, which is not a square as $q \notin GP_2F$. It is well-known (and easy to show) that $q_{F(\sqrt{d})} \in GP_2F(\sqrt{d})$ is anisotropic. This yields $m_{ext}(q) = 2$.

(ii), (iv) If $\dim q = 5$ (resp. $\dim q = 9$), then $m_{ext}(q) = 3$ (resp. 4) follows from Lemma 2.3, and it is clear that $m(q) \geq 4$ (resp. $m(q) \geq 5$) as we assumed q not to be a Pfister neighbor.

(iii) If $6 \leq \dim q \leq 8$, then it was shown in [Lag, Corollary 2] that $m_{ext}(q) = 3$ iff q does not contain an Albert form as subform. For these dimensions, it is also clear that $m(q) \geq 4$.

If q contains a subform in GP_2F, then we can write q in the form $q = q_0 \perp \mu$ where q_0 is a 5-dimensional Pfister neighbor. Proposition 2.7 shows that either $m(q) = \infty$ or $m(q) \leq m(q_0) + \dim \mu = 3 + (\dim q - 5) = \dim q - 2$. In both cases, we have $m(q) \neq \dim q - 1$. □

Remark 2.10. If $\dim q \leq 6$ then the results of Fitzgerald ([Fi2]) show that $m(q) = \infty$ if and only if $W(F(q)/F) = 0$. In other words, $m(q)$ is finite iff q is conservative.

Example 2.11. In general, if K/F is an extension such that q_K is anisotropic, then it is possible that $m_{ptr}(q_K) > m_{ptr}(q)$ and $m_{ext}(q_K) > m_{ext}(q)$ provided K/F is not purely transcendental (cf. Lemma 2.2).

Let $F = \mathbb{Q}((t))$ and consider $q = \langle 1, 1, 1, 2\rangle \perp t\langle 1, -3\rangle$. It follows from Springer's theorem [Lam, Chapter VI, § 1] that q is anisotropic and does not contain a subform similar to $\langle 1, -6\rangle$. In particular, for $K = F(\sqrt{6}) = L((t))$ with $L = \mathbb{Q}(\sqrt{6})$, we have that q_K is anisotropic by [Lam, Chapter VII, Lemma 3.1]. Furthermore, $q_K \in I^2K$ as $d_{\pm}q = 6 \in F^*/F^{*2}$, i.e. q_K is an Albert form. It follows from Proposition 2.9 that $m_{ext}(q) = 3 < 4 = m_{ext}(q_K)$, $m_{ptr}(q) = 4$ and $m_{ptr}(q_K) \in \{4, 5\}$.

Suppose $m_{ptr}(q_K) = 4$. Then by Theorem 2.6, $m(q_K) = 4$ and there exists an anisotropic $\pi \in GP_4K$ such that $q_K \subset \pi$. Now K is formally real and q_K is totally indefinite (it contains the subform $t\langle 1, -3\rangle$ which has total signature zero). Hence, necessarily π must also be totally indefinite. Since Pfister forms are either definite or have signature zero with respect to any given ordering, it follows that π has total signature zero, i.e. π is a torsion form by Pfister's local-global principle (see, e.g., [Lam, Chapter VIII, Theorem 4.1]). Since L is a global field, I^3L is torsion free, thus I^4K is torsion free, again by Springer's theorem. Hence, π is hyperbolic, a contradiction. We conclude that $m_{ptr}(q_K) = 5 > 4 = m_{ptr}(q)$.

Remark 2.12. The following has been pointed out by Bruno Kahn. Let F be a field of cohomological dimension ≤ 3. Then there are no anisotropic 4-fold Pfister forms over F, and if q is an anisotropic Albert form over F, this implies that there do not exist purely transcendental extensions of F over which q becomes 4-embeddable.

3. STABLE PFISTER NEIGHBORS

Definition 3.1 Let q be an anisotropic form over a field F and let n be the integer for which $2^{n-1} < \dim q \leq 2^n$. We say that q is *stably* a Pfister neighbor or that q is a *stable Pfister neighbor* if $m_{ext}(q) = n$. We say that q is *nonstably* a Pfister neighbor or that q is a *nonstable Pfister neighbor* if $m_{ext}(q) = n + 1$.

Let q be any form over any field F. If K is a field over which q attains the maximal possible Witt index (i.e. the anisotropic part of q_K is of dimension ≤ 1), then q_K is a Pfister neighbor of a hyperbolic Pfister form. The terminology 'stable/nonstable' refers to whether it is possible not only to find an extension over which an anisotropic q becomes a Pfister neighbor, but over which q also stays anisotropic.

Remark 3.2. Let q be an anisotropic form over F of dimension $2^{n-1} + r$, $1 \leq r \leq 2^{n-1}$. It was shown in [Ho, Corollary 1] that the Witt index $i_W(q_{F(q)})$ of q over its own function field $F(q)$ is $\leq r$. If it is equal to r, we also say that q has *maximal splitting*. Note that q has always maximal splitting if $r = 1$ or if q is a Pfister neighbor. It was shown in [Ho, Corollary 3 (i)] that if q as above has maximal splitting, then q is stably a Pfister neighbor.

The proof of the following observations is an easy application of the definitions and Theorem 2.6. We leave the details to the reader.

Proposition 3.3. *Let q be an anisotropic form over F with $2^{n-1} < \dim q \leq 2^n$.*

(i) *q is a Pfister neighbor iff $m_{ptr}(q) = n$ iff $m(q) = n$.*

(ii) *q is not a Pfister neighbor but stably a Pfister neighbor iff $m_{ptr}(q) = m_{ext}(q) + 1 = n + 1$.*

(iii) *q is not stably a Pfister neighbor iff $m_{ext}(q) = n + 1$.*

In particular, suppose that q has maximal splitting. Then $m(q) = m_{ptr}(q) = m_{ext}(q) = n$ iff q is a Pfister neighbor, and otherwise $m_{ext}(q) = n$ and $m(q) \geq m_{ptr}(q) = n + 1$.

Corollary 3.4. *Let q be an anisotropic form over a formally real field F such that $2^{n-1} < \dim q \leq 2^n$, such that q is not a Pfister neighbor, and such that q is definite with respect to some ordering α of F. Then q is stably a Pfister neighbor and $m_{ptr}(q) = m_{ext}(q) + 1 = n + 1$.*

Proof. We may assume that q is positive definite at the ordering α. Let $K = F_\alpha$ be a real closure of F with respect to α. Then $q_K \cong \langle 1, \cdots, 1\rangle$ is a subform of

$\langle\langle -1, \cdots, -1 \rangle\rangle \in P_n K$. Hence, q is stably a Pfister neighbor, and since it is not a Pfister neighbor by assumption, $m_{ptr}(q) = m_{ext}(q) + 1 = n + 1$ by the previous proposition. $\square$

Definition 3.5. Let d be a positive integer. We define

$$M(d) \quad = \quad \{m \mid \text{ there exist a field } F \text{ and an anisotropic form } q \text{ over } F, \dim q = d, \text{such that } m(q) = m\}.$$

Note that $M(d)$ can contain positive integers as well as ∞ as possible values.

Theorem 3.6. *Let d be a positive integer and n be such that $2^{n-1} < d \leq 2^n$. Then $n \leq d - 1$ and*

(i) *if $1 \leq d \leq 3$ then $M(d) = \{n\} = \{d - 1\}$;*

(ii) *if $d \geq 4$, then $M(d) = \{n, \cdots, d - 1\} \cup \{\infty\}$.*

Moreover, for all $m \in M(d)$ there exist a field F and an anisotropic d-dimensional stable Pfister neighbor q over F such that $m(q) = m$.

Remark 3.7. In the above theorem, (i) is clear as forms of dimension ≤ 3 are always Pfister neighbors. It should also be noted that anisotropic 4-dimensional forms are always stable Pfister neighbors, cf. Proposition 2.9 (i).

For the proof of this theorem, we need the following Lemma.

Lemma 3.8. *Let q_0 be an anisotropic form over F and let $q = q_0 \perp \langle t \rangle$ over $E = F((t))$. Then the following holds.*

(i) *Either $m(q) = m(q_0) = \infty$, or $m(q) = m(q_0) + 1 < \infty$.*

(ii) *If q_0 is stably a Pfister neighbor, then so is q. In particular, if $\dim q_0$ is not a 2-power, then $m_{ext}(q) = m_{ext}(q_0)$.*

Proof. (i) Let $m_0 = m(q_0)$ and $m = m(q)$. If $m_0 < \infty$, let $\pi \in GP_{m_0} F$ be anisotropic such that $q_0 \subset \pi$. Then $q \subset \pi \otimes \langle\langle -at \rangle\rangle \in GP_{m_0+1} E$ for any $a \in F^*$ represented by q_0. By Springer's theorem, $\pi \otimes \langle\langle -at \rangle\rangle$ is anisotropic. Hence $m \leq m_0 + 1 < \infty$.

If $m < \infty$, let $\tau \in GP_m E$ be such that $q \subset \tau$. It follows again readily from Springer's theorem that $\tau \simeq \pi \perp at\pi$ for some $\pi \in GP_{m-1} F$, that $q_0 \subset \pi$, and that 1 is represented by $a\pi$ over F. In particular, $m_0 \leq m - 1 < \infty$.

The above shows that either both m and m_0 are infinite, or both are finite and $m = m_0 + 1$.

(ii) Clearly, $m_{ext}(q_0) \leq m_{ext}(q)$, and if $\dim q_0$ is a 2-power, then q is stably a Pfister neighbor by Lemma 2.3 (independently of q_0 being stably a Pfister neighbor or not). Thus, we may assume that $2^{n-1} < \dim q_0 < 2^n$ for some integer n. Then, again by Lemma 2.3, we have $m_{ext}(q_0), m_{ext}(q) \in \{n, n + 1\}$, and it suffices to show that if $m_{ext}(q_0) = n$ then $m_{ext}(q) = n$.

So let q_0 be stably a Pfister neighbor. Then there exists a field extension K/F and an anisotropic $\pi \in GP_n K$ such that $(q_0)_K \subset \pi$. We may assume K to be linearly disjoint from $E = F((t))$ over F, so that $K((t))$ is a power series field in the same variable t over K. Since $\dim q_0 < 2^n$, there exists an $a \in K^*$ such that $(q_0)_K \perp \langle a \rangle \subset \pi$. Let $L = K((t))(\sqrt{at})$. Since π is defined over K and anisotropic, and since L/K is again a complete field with residue field K, it follows that π_L is anisotropic. Also, $\langle a \rangle_L \cong \langle t \rangle_L$ and thus $q_L \subset \pi_L$, which shows that q is stably a Pfister neighbor. □

A repeated application of the previous lemma yields

Corollary 3.9 (cf. Proposition 2.7). *Let F be a field with an anisotropic Pfister neighbor q_0. Let $E = F((t_1)) \cdots ((t_r))$ and $\mu = \langle t_1, \cdots, t_r \rangle$. Let $q = (q_0)_E \perp \mu$. Then q is stably a Pfister neighbor and $m(q) = m(q_0) + \dim \mu$.*

Proof of Theorem 3.6 (ii). Let $d \geq 4$ and n be positive integers such that $2^{n-1} < d \leq 2^n$. By Remark 2.8, it is clear that $M(d) \subset \{n, \cdots, d-1\} \cup \{\infty\}$.

Let F be any field with an anisotropic n-fold Pfister form. Any d-dimensional Pfister neighbor q of this Pfister form will have $m(q) = n$. Thus, $n \in M(d)$. To show that $\{n, \cdots, d-1\} \subset M(d)$ and that these values can be realized by stable Pfister neighbors, we use induction on d and note that this is true for $d = 3$ by part (i).

So let $d \geq 4$. By what was mentioned above, it suffices to show that the values $\{n+1, \cdots, d-1\}$ can be realized by stable Pfister neighbors. Note that $d \geq 4$ and $2^{n-1} < d \leq 2^n$ implies $n \leq d-2$. By induction hypothesis, all the values $k \in \{n, \cdots, d-2\}$ can be realized as values $m(q'_k)$ of stable Pfister neighbors q'_k of dimension $d-1$ over suitable fields F_k. By passing to the forms $q_k = q'_{k-1} \perp \langle t \rangle$ over $F_{k-1}((t))$ and applying Lemma 3.8 we conclude that the values $k \in \{n+1, \cdots, d-1\}$ are realized by the d-dimensional stable Pfister neighbors q_k.

To realize the value ∞ in the case $d \geq 4$, let F be any field such that the torsion part of I^3F is 0 and such that there exists an anisotropic 4-dimensional form φ over F such that $d_{\pm}\varphi \neq 1 \in F^*/F^{*2}$ (i.e. $\varphi \notin GP_2F$) and such that φ is torsion in WF. Since Pfister forms over F containing a subform similar to φ will be torsion as φ is torsion, and in I^3F as $\varphi \notin GP_2F$, they will be hyperbolic. This implies that $m(\varphi) = \infty$. Furthermore, φ is stably a Pfister neighbor by Proposition 2.9 (i). For $d \geq 5$, let $q = \varphi \perp \langle t_1, \cdots, t_{d-4} \rangle$ over $E = F((t_1)) \cdots ((t_{d-4}))$. By Lemma 3.8, q is a d-dimensional stable Pfister neighbor with $m(q) = \infty$. □

Remark 3.10. The form φ in the last part of the previous proof can, for instance, be realized over $F = \mathbb{Q}$. We have that $I^3\mathbb{Q}$ is torsion-free. Let $\varphi = \langle 1, 1, -7, -14 \rangle$. This form has signature zero and is therefore torsion. On the other hand, by passing to $\mathbb{Q}_7$ one checks readily that φ is anisotropic. Also, $d_{\pm}\varphi = 2 \notin \mathbb{Q}^{*2}$. Hence, we have $m(\varphi) = \infty$.

4. NONSTABLE PFISTER NEIGHBORS

Having determined all values $m(q)$ which can be realized by stable Pfister neighbors q in a given dimension over suitable fields, the natural question to ask is which values can be realized by nonstable Pfister neighbors. This problem seems to be more difficult as we lack nice general criteria by which one could decide whether a form is a nonstable Pfister neighbor or not. In the sequel, we will prove some partial results.

Definition 4.1. Let d be a positive integer. We define

$$M_{ns}(d) = \{m \mid \text{there exist a field } F \text{ and an anisotropic form } q \text{ over } F, \dim q = d, \text{such that } q \text{ is a nonstable Pfister neighbor and } m(q) = m\}.$$

The following lemma is rather trivial (the first part of it having been mentioned in Remarks 3.2 and 3.7).

Lemma 4.2. (i) *If $d \leq 5$ or if d is of the form $2^n + 1$ for some $n \geq 0$, then $M_{ns}(d) = \emptyset$.*

(ii) *Let $n \geq 2$ and $2^n + 2 \leq d \leq 2^{n+1}$. Then $M_{ns}(d) \subset \{n+2, \cdots, d-1, \infty\}$.*

The obvious question is whether one has equality in part (ii).

Another trivial but quite useful observation is the following.

Lemma 4.3. *Let $q_0 \subset q$ be anisotropic forms over F with $2^n + 2 \leq \dim q_0 \leq \dim q \leq 2^{n+1}$ for some $n \geq 2$. If q_0 is a nonstable Pfister neighbor, then q is a nonstable Pfister neighbor.*

Lemma 4.4. *Let n, d, and d_0 be integers such that $2^n + 2 \leq d_0 < d \leq 2^{n+1}$. Then for finite $m \in M_{ns}(d_0)$ and $s \in \{0, 1, \ldots, d - d_0\}$, we have $m + s \in M_{ns}(d)$. Moreover, $M_{ns}(d_0) \subset M_{ns}(d)$.*

Proof. Since $m \in M_{ns}(d_0)$, there exists a d_0-dimensional nonstable Pfister neighbor q_0 over a suitable field F such that $m = m(q_0)$. Let $\pi \in GP_mF$ be an anisotropic form such that $q_0 \subset \pi$. Since $d_0 \leq d - s \leq 2^{n+1} \leq 2^m$, there exists a $(d-s)$-dimensional form q_0' such that $q_0 \subset q_0' \subset \pi$. Since $m = m(q_0) \leq m(q_0') \leq m(\pi) = m$, we have $m(q_0') = m$. Now let $E = F((t_1)) \cdots ((t_s))$ and $q = (q_0')_E \perp \langle t_1, \cdots, t_s \rangle$. Lemma 3.8 (i) shows that $m(q) = m(q_0') + s = m + s$. By Lemma 4.3, q is a nonstable Pfister neighbor. Since $\dim q = d$, it follows that $m + s \in M_{ns}(d)$. Setting $s = 0$, we get $m \in M_{ns}(d)$. If $m = \infty \in M_{ns}(d_0)$, then Lemmas 3.8 (i) and 4.3 show that $m = \infty \in M_{ns}(d)$. $\square$

Corollary 4.5. *Let n, d, and d_0 be integers such that $2^n + 2 \leq d_0 < d \leq 2^{n+1}$. Let $s < r$ be such that $\{s, s+1, \ldots, r, \infty\} \subset M_{ns}(d_0)$. Then $\{s, s+1, \ldots, r + d - d_0, \infty\} \subset M_{ns}(d)$.*

Some of our constructions will be based on the following result (cf. [Ho, Proposition 1]).

Lemma 4.6. *Let $n \geq 2$ be an integer, $\varphi \cong \langle 1 \rangle \perp \varphi' \in P_nF$ and $\psi \cong \langle 1 \rangle \perp \psi' \in$*

P_2F be anisotropic such that $\varphi' \perp -\psi'$ is anisotropic. Then $\varphi' \perp -\psi'$ is not stably a Pfister neighbor, i.e. $m_{ext}(\varphi' \perp -\psi') = n+2$. In particular, $M_{ns}(2^n+2) \neq \emptyset$.

It seems worth noting the following immediate consequence of Lemmas 4.3 and 4.6.

Corollary 4.7. *Let $d > 0$ be an integer. Then there exists a field F with an anisotropic form of dimension d which is not stably a Pfister neighbor if and only if there exists an integer $n \geq 2$ such that $2^n + 2 \leq d \leq 2^{n+1}$.*

Proposition 4.8. *Let $n \geq 2$ and d be integers with $2^n + 2 \leq d \leq 2^{n+1}$. Then*

$$\{n+2, n+3, \infty\} \subset \{n+2, \cdots, n+d-2^n+1, \infty\} \subset M_{ns}(d).$$

Proof. Corollary 4.5 shows that it suffices to verify that $\{n+2, n+3, \infty\} \subset M_{ns}(d_0)$ where $d_0 = 2^n + 2$. Let F be a field with an anisotropic Pfister form $\varphi \cong \langle 1 \rangle \perp \varphi' \in P_nF$. Let $E = F((x))((y))$ and $\psi = \langle\langle x, y \rangle\rangle = \langle 1 \rangle \perp \langle -x, -y, xy \rangle = \langle 1 \rangle \perp \psi'$. Consider now the (2^n+2)-dimensional form

$$q = \varphi' \perp -\psi' = \varphi' \perp \langle x, y, -xy \rangle.$$

By Springer's theorem, q is clearly anisotropic. By Lemma 4.6, q is is not stably a Pfister neighbor.

Now suppose that there exists some anisotropic form $\pi \in GPE$ such that $q \subset \pi$. Then $\varphi' \subset \pi$. Since π becomes isotropic and hence hyperbolic over $E(\varphi')$, π becomes hyperbolic over $E(\varphi)$ as φ' is a Pfister neighbor of the Pfister form φ. Let $a \in E^*$ be any element represented by φ' and thus by φ and π. By the Cassels-Pfister subform theorem, $a\varphi \subset a\pi$ and thus $\varphi \subset \pi$, so that we can write $\pi \cong \varphi \perp \gamma$ for some form γ over E. (Note that π represents 1 as φ does, so in fact $\pi \in PE$.) Now π represents x because q does, but φ does not represent x, and Springer's theorem yields that necessarily x is represented by γ, so that $\varphi \perp \langle x \rangle \subset \pi$. By a similar reasoning as before, $\varphi \otimes \langle 1, x \rangle \subset \pi$ as $\varphi \perp \langle x \rangle$ is a Pfister neighbor of $\varphi \otimes \langle 1, x \rangle \in P_{n+1}E$. Repeating this once more for y, one gets that $\varphi \otimes \langle 1, x, y, xy \rangle \subset \pi$. Note that $\varphi \otimes \langle 1, x, y, xy \rangle$ is certainly anisotropic. This shows that $m(q) \geq n+2$. Let us consider three cases.

Case 1. Suppose that φ represents -1. Then it follows readily that $q \subset \varphi \otimes \langle 1, x, y, xy \rangle$. Hence, $m(q) = n+2$. An example of this type is given by $F = \mathbb{C}(x_1, \cdots, x_n)$, the rational function field in n variables over the complex numbers, and $\varphi = \langle\langle x_1, \cdots, x_n \rangle\rangle$.

Case 2. Suppose that φ does not represent -1. Then $q \not\subset \varphi \otimes \langle 1, x, y, xy \rangle$ and it follows that $m(q) \geq n+3$. If there exists an $a \in F^*$ such that $\varphi \otimes \langle\langle a \rangle\rangle$ is anisotropic and represents -1, then $m(q) = n+3$ as $q \subset \varphi \otimes \langle\langle a \rangle\rangle \otimes \langle 1, x, y, xy \rangle$. This situation is realized over $\mathbb{Q}(x_1, \cdots, x_{n-1})$ for $\varphi = \langle\langle 3, x_1, \cdots, x_{n-1} \rangle\rangle$ and $a = -1$.

Case 3. Suppose that there does not exist an anisotropic form σ over F representing -1 and containing φ as a subform. Then $xy\varphi \subset \pi$ and $\langle -xy \rangle \subset \pi$ together with Springer's theorem yield a contradiction, so in this case

$m(q) = \infty$. An example where this happens is provided by $F = \mathbb{R}$ and $\varphi = \langle\langle -1, \cdots, -1 \rangle\rangle$. $\square$

Remark 4.9. (i) A slight modification of this construction can be used to show that the converse of Lemma 4.3 is generally false. In fact, let F be a field with an anisotropic $\varphi = \langle 1 \rangle \perp \varphi' \in P_n F$, $n \geq 2$, and let $\rho = \langle 1 \rangle \perp \rho' \subset \varphi$, $\dim \rho = 3$. Consider $q = \varphi' \perp x\rho \perp \langle y, -xy \rangle$. Then $\dim q = 2^n + 4$ and q is not stably a Pfister neighbor since it contains the nonstable Pfister neighbor $\varphi' \perp -\langle\langle x, y \rangle\rangle'$. On the other hand, q contains the subform $\varphi' \perp x\rho$ which is in fact a Pfister neighbor of $\varphi \otimes \langle 1, x \rangle \in P_{n+1} F((x))((y))$ of dimension $2^n + 2$.

(ii) In the second case of the above proof, where φ does not represent -1, the existence of an a as in the proof can always be shown provided φ is a torsion form. For in that case, let k be maximal such that $\sigma_k \otimes \varphi$ is anisotropic for $\sigma_k = \langle\langle -1, \cdots, -1 \rangle\rangle \in P_k F$ (such a k exists as φ is torsion). One has $k \geq 2$ as by assumption the Pfister neighbor $\varphi \perp \langle 1 \rangle$ of $\varphi \otimes \sigma_1$ is anisotropic. We have that $\sigma_{k+1} \otimes \varphi$ is then isotropic and hence hyperbolic. Thus, its Pfister neighbor $\sigma_k \otimes \varphi \perp \langle 1 \rangle$ is isotropic and there exist $x_i \in F$, $1 \leq i \leq 2^k$, represented by φ, such that $-1 = \sum_{i=1}^{2^k} x_i$. Let $u = \sum_{i=2}^{2^k} x_i$. Then $\varphi \perp \langle u \rangle$ represents -1 and is anisotropic as it is a subform of $\sigma_k \otimes \varphi$. Hence $\varphi \otimes \langle 1, u \rangle \in P_{n+1} F$ is anisotropic and represents -1.

For some d, we were able to prove that equality holds in Lemma 4.2. We will make use of the following criterion.

Lemma 4.10. *Let $n \geq m \geq 2$ be integers. Let q be an anisotropic form over F such that $q \in I^m F$ and $2^{n+1} > \dim q > 2^{n+1} - 2^m$, then q is a nonstable Pfister neighbor.*

Proof. Suppose q a stable Pfister neighbor. Then there exist a field extension E/F, an anisotropic form $\pi \in GP_{n+1} E$ and a form γ over E such that $q_E \perp \gamma \cong \pi$. We have that γ is anisotropic, $0 < \dim \gamma < 2^m$, and $\gamma \in I^m E$ as $q_E \in I^m E$, but this contradicts the Arason-Pfister Hauptsatz. Hence q is a nonstable Pfister neighbor. $\square$

Proposition 4.11. *Let $n \geq 2$ and d be integers such that $2^{n+1} - 2 \leq d \leq 2^{n+1}$. Then $M_{ns}(d) = \{n+2, \cdots, d-1, \infty\}$.*

Proof. By Corollary 4.5, it suffices to consider the case $d = 2^{n+1} - 2$. Let $1 \leq e \leq 2^n$, let $m \geq 0$ be such that $2^{m-1} < e \leq 2^m$, let $r = 2^{n+1} - 3 - e$.

Let F be a field with an anisotropic $\varphi \in P_m F$ which does not represent -1 but such that there exists an $a \in F^*$ such that $\varphi \otimes \langle\langle a \rangle\rangle$ is anisotropic and represents -1 (see the proof of Proposition 4.8 and Remark 4.9). Let ρ be a subform of φ of dimension e. Note that ρ is a Pfister neighbor of φ. Let $b = \det \rho$.

Let now $E = F((x_1)) \cdots ((x_r))$ and put $q = \rho \perp \langle x_1, \cdots, x_r, -bx_1x_2 \cdots x_r \rangle$. Clearly, q is anisotropic. Furthermore, $\dim q = 2^{n+1} - 2$ and $\det q = -1$, in particular $q \in I^2 E$. By Lemma 4.10, q is a nonstable Pfister neighbor. We also note that $b = \det \rho$ is represented by φ as φ is multiplicative and $\rho \subset \varphi$. Since -1 and b are represented by $\varphi \otimes \langle\langle a \rangle\rangle$ which is also multiplicative, we have that $-b$ is represented by $\varphi \otimes \langle\langle a \rangle\rangle$.

One verifies now readily that the form $\pi = \varphi \otimes \langle 1, x_1 \rangle \otimes \cdots \langle 1, x_r \rangle \otimes \langle\langle a \rangle\rangle$ is anisotropic and that $q \subset \pi$. In particular, $m(q) \leq m + r + 1$. A reasoning similar to that in the proof of Proposition 4.8 (in particular case 2 in that proof) shows that $m(q) = m + r + 1 = 2^{n+1} - 2 + m - e$.

For $e = 1$ we have $m = 0$ and thus $m(q) = (2^{n+1} - 2) - 1$. As e increases by 1, m stays the same or increases also by 1, so that $m(q)$ decreases at most by 1. For $e = 2^n$, we have $m = n$ and $m(q) = (2^{n+1} - 2) + n - 2^n$.

As a consequence, we see that for $d = 2^{n+1} - 2$, we can realize each value in $\{d + n - 2^n, \cdots, d - 1\}$ as $m(q)$ of a nonstable Pfister neighbor q of dimension d over a suitable field. We already know by Proposition 4.8 that all values in $\{n + 2, \cdots, d + n - 2^n + 1, \infty\}$ can be realized. This completes the proof. □

Question 4.12. What is the set $M_{ns}(d)$ for $2^n + 2 \leq d \leq 2^{n+1} - 3$, $n \geq 3$?

We were able to obtain values other than those in Proposition 4.8 for certain d with $2^n + 2 \leq d \leq 2^{n+1} - 3$, $n \geq 3$. But we do not know whether the list of values obtained for such d is complete or not (cf. also the list of values which cannot be ruled out, Lemma 4.2 (ii)). We refrain from giving the rather technical details of these constructions.

5. AN APPLICATION TO MILNOR'S K-GROUPS

Let $K_n^M F$ denote the Milnor K-group of a field F in degree n, let $k_n F$ be the quotient group $K_n^M F / 2K_n^M F$, and denote the graded ring $\bigoplus_{i=0}^{\infty} k_i(F)$ by $k_*(F)$. Recall that as an associative ring with unit, $k_* F$ is generated by symbols $\{a\}$, $a \in F^*$ subject to the defining relations $\{ab\} = \{a\} + \{b\}$, $\{a\}\{1 - a\} = 0$ for all $a \in F^*$, $a \neq 1$, and $2\{a\} = \{a\} + \{a\} = 0$. An element of type $\{a_1\}\{a_2\} \cdots \{a_n\}$ will be called a symbol (of degree n), and we will write $\{a_1, \cdots, a_n\}$ for short.

Let L/F be a field extension. Since the natural map $k_*(F) \to k_*(L)$ is a graded ring homomorphism, its kernel is a homogeneous ideal of $k_*(F)$. We will denote this kernel by $k_*(L/F)$ and use the notation $k_n(L/F)$ to denote the kernel of the natural group homomorphism $k_n(F) \to k_n(L)$. By a system of generators of a homogeneous ideal in $k_*(F)$ we always mean a system of generators consisting of homogeneous elements.

As a general reference for any undefined terminology we might use and for some basic results, we refer to [Mi].

In this section, we want to look at the kernel of the homomorphism $k_*(F) \to k_*(F(q))$ where q is a quadratic form over F and F is of characteristic $\neq 2$. It has been conjectured that this kernel can be generated by symbols (cf. [Vi]). Another question one can ask is the following: Suppose one has a system of homogeneous generators of this kernel, what can one say about the degrees of these generators? The following theorem shows that a simple answer to this question seems highly unlikely.

Theorem 5.1. *Let $d > 3$ and n be such that $2^{n-1} < d \leq 2^n$. Then there exists a field E and a d-dimensional form q over E such that each system of homogeneous generators of the ideal*

$$k_*(E(q)/E) = \ker[k_*(E) \to k_*(E(q))]$$

in the ring $k_*(E)$ contains at least one element of degree $\leq n+1$ and at least one element of degree $d-1$.

For the proof, we need the following.

Lemma 5.2. *Let F_0 be a field of characteristic $\neq 2$ and let $q = \langle 1, -t_1, \cdots, -t_r \rangle$ over $F = F_0((t_1)) \cdots ((t_r))$. Then the following holds.*

(i) *If $i < r$ then $k_i(F(q)/F) = 0$. If $i \geq r$ then*

$$k_i(F(q)/F) = \{t_1, \cdots, t_r\} \cdot k_{i-r}(F).$$

In other words, $k_*(F(q)/F)$ is generated by the element $\{t_1, \cdots, t_r\}$.

(ii) *Let E/F be a purely transcendental extension. Then any system of homogeneous generators of the ideal $k_*(E(q)/E)$ contains at least one element of degree r.*

Proof. (i) Since $q \subset \langle\langle t_1, \cdots, t_r \rangle\rangle$, the r-fold Pfister form $\langle\langle t_1, \cdots, t_r \rangle\rangle$ becomes hyperbolic over $F(q)$. Hence, $\{t_1, \cdots, t_r\} \in k_r(F(q)/F)$ (cf. [EL, Main Theorem 3.2]). It remains to show that $k_*(F(q)/F) \subset \{t_1, \cdots, t_r\} \cdot k_*(F)$.

It is well-known that $k_*(F)$ is a graded $k_*(F_0)$-module freely generated by the elements $\{t_{i_1}, \cdots, t_{i_k}\}$ where $1 \leq i_1 < \cdots < i_k \leq r$ and $k \geq 0$. Identifying $F(\sqrt{t_i})$ with $F_0((t_1)) \cdots ((t_i')) \cdots ((t_r))$ where $t_i' = \sqrt{t_i}$, we obviously have $\{t_i\} \cdot k_*(F) = k_*(F(\sqrt{t_i})/F)$. Hence,

$$\{t_1, \cdots, t_r\} \cdot k_*(F) = \bigcap_{i=1}^{r} \{t_i\} \cdot k_*(F) = \bigcap_{i=1}^{r} k_*(F(\sqrt{t_i})/F).$$

Since $q_{F(\sqrt{t_i})}$ is isotropic ($i = 1, \ldots, r$), we have that $F(\sqrt{t_i})(q)/F(\sqrt{t_i})$ is purely transcendental, which implies that $k_*(F(q)/F) \subset k_*(F(\sqrt{t_i})/F)$ for all $i = 1, \cdots, r$. Hence $k_*(F(q)/F) \subset \{t_1, \cdots, t_r\} \cdot k_*(F)$.

(ii) If $E = F$, then (ii) is obvious in view of (i). To prove (ii) in the general case, it suffices to notice that there exists a specialization homomorphism of graded rings $\psi : k_*(E) \to k_*(F)$, which maps the ideal $k_*(E(q)/E)$ onto the ideal $k_*(F(q)/F)$. (If $E = F(x)$ is the rational function field in one variable, this specialization homomorphism is essentially the homomorphism defined in [Mi, Lemma 2.2] for the prime element x. For $E = F(x_1, \cdots, x_m)$, this homomorphism is obtained simply by composition of specialization homomorphisms of that type.)

In fact, suppose there exists a system of homogeneous generators of $k_*(E(q)/E)$ which are all of degree $\neq r$. Since homogeneous elements in $k_*(F(q)/F)$ are all of degree $\geq r$ by (i), all generators of degree $< r$ map to zero under ψ. The other generators of degree $> r$ map to elements of degree $> r$, thus the homogeneous elements in the image would all be of degree $> r$, a contradiction because $\{t_1, \cdots, t_r\}$ is of degree r and in the image of ψ. $\square$

Proof of Theorem 5.1. Let $r = d - 1$ and let q and F be as in Lemma 5.2. By Corollary 3.9, we have $m(q) = d - 1$ and $m_{ext}(q) = n$. Since $d > 3$, we have $d \geq n + 2$. Hence $m_{ptr}(q) = \min\{m(q), m_{ext}(q) + 1\} = \min\{d - 1, n + 1\} = n + 1$. This means that there exists a purely transcendental extension E/F and an anisotropic form $\pi = \langle\langle a_1, \cdots, a_{n+1}\rangle\rangle \in P_{n+1}(E)$ such that $q \subset \pi$.

Now π is anisotropic and hence $\{a_1, \ldots, a_{n+1}\} \neq 0$. Since $q \subset \pi$, we have that $\pi_{E(q)}$ is isotropic. Therefore, $\{a_1, \cdots, a_{n+1}\} \in k_{n+1}(E(q)/E)$. Thus, we have proved that $k_*(E(q)/E)$ contains a nontrivial element of degree $n + 1$. To complete the proof it suffices to invoke Lemma 5.2 (ii). $\square$

Remark 5.3. (i) Recent results by Voevodsky [Vo] and by Orlov-Vishik-Voevodsky [OVV] show that the canonical homomorphisms $k_nF \to H^n_{et}(F, \mathbb{Z}/2\mathbb{Z})$ resp. $k_nF \to I^nF/I^{n+1}F$ mapping $\{a_1, \cdots, a_n\}$ to the n-fold cup product $(a_1, \cdots, a_n) = (a_1) \cup \cdots \cup (a_n)$ resp. to $\langle\langle a_1, \cdots, a_n\rangle\rangle \bmod I^{n+1}F$ are isomorphisms for fields F of characteristic $\neq 2$. Thus, we can exchange the functor $k_*(\,\cdot\,)$ in the formulation of Theorem 5.1 by the cohomology functor $H^*_{et}(\,\cdot\,, \mathbb{Z}/2\mathbb{Z})$.

(ii) Modulo results in [OVV], one can also replace '$\leq n + 1$' in the statement of Theorem 5.1 by '$= n + 1$'. Let us give the argument. It was shown in [OVV] that if φ is a form over a field L and $\dim \varphi > 2^{n-1}$, then $k_i(L(\varphi)/L) = 0$ for all $i \leq n - 1$. This shows in particular that with the notations as in the theorem above, $k_i(E(q)/E) = 0$ for all $i \leq n - 1$. Thus, it remains to show that $k_n(E(q)/E) = 0$.

Now the E from the theorem is a purely transcendental extension of the field F in Lemma 5.2, and the form q is also defined as in that lemma. We claim that $k_n(E(q)/E) = k_n(F(q)/F)$. Once we have this, it follows from Lemma 5.2 (i) that $k_n(E(q)/E) = 0$.

By induction on the transcendence degree, it suffices to consider the case $E = F(t)$. Using the same notations as in Lemma 2.5 and using the fact that $F(q)(t) = E(q)$, we get in a similar way the following commutative diagram whose upper row is exact and whose lower row is exact at $k_nF(q)$ (see also [Mi, Theorem 2.3]):

$$\begin{array}{ccccccccc}
0 & \longrightarrow & k_nF & \xrightarrow{i} & k_nE & \xrightarrow{\oplus_p \partial_p} & \bigoplus_p k_{n-1}\overline{E}_p & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & k_nF(q) & \xrightarrow{i} & k_nE(q) & \xrightarrow{\oplus_p \partial_p} & \bigoplus_p k_{n-1}\overline{E}_p(q) & &
\end{array}$$

This diagram gives rise to the exact sequence

$$0 \longrightarrow k_n(F(q)/F) \xrightarrow{i} k_n(E(q)/E) \xrightarrow{\oplus_p \partial_p} \bigoplus_p k_{n-1}(\overline{E}_p(q)/\overline{E}_p).$$

The last term in this sequence is 0 by the result in [OVV] mentioned above. Thus, $k_n(E(q)/E) = k_n(F(q)/F)$ as claimed.

It should be remarked that the results from [Vo] and [OVV] to which we refer above have not yet been published in refereed journals at the time the present paper was written.

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(Received March 1999)